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LEAST COMMON MULTIPLE OF PATH WITH 3 EDGES AND CARTESIAN PRODUCT OF CYCLES

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Abstract

A graph G without isolated vertices is a least common multiple of two graphs H1 and H2 if G is a smallest graph in terms of number of edges such that there exists a decomposition of G into edge disjoint copies of H_1 and there exists a decomposition of G into edge disjoint copies of H_2. The concept was introduced by G.Chartrand et.al and proved that every two non empty graphs have a least common multiple. Finding least common multiple of different pairs of graphs is a topic of prime interest in this area. In this paper we found least common multiple of P_4 and cartesian product of C_m and C_n, $C_m \times C_n$. This is a beginning in this area since least common multiple of graphs where at least one of them is a general product graph has not been found in literature.

Keywords **:** *Graph Decomposition, Least Common Multiple*
MSC2020: 05C38, 05C51, 05C70

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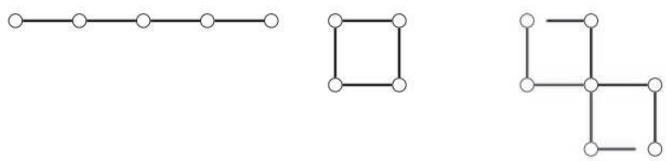
1. Introduction

A non-empty graph G is decomposable into the subgraphs $G_1, G_2, \ldots, G_n$ of G if no G_i for i = 1, 2, ... , n has isolated vertices and the edge set E(G) of G can be partitioned into the subsets $E(G_1), E(G_2), \ldots, E(G_n)$. If $G_i \cong H$ for each 'i', then we say that H divides G and write $H|G$. A graph H without isolated vertices is a least common multiple of two graphs G_1 and G_2 if H is a graph of minimum size such that $G_1|H$ and $G_2|H$. The size of a least common multiple of G_1 and G_2 is denoted by $lcm(G_1,G_2)$. This concept was introduced by G.Chartrand et.al [2] and they proved that every two graphs have a least common multiple. But least common multiple of two graphs need not be unique. For example, P_7 and C_6 are least common multiples of P_4 and P_3, where P_n denote the path on 'n' vertices. Finding least common multiples of pairs of graphs is a topic of prime interest in this area. The Number theoretic *lcm* need not always be same as the graph theoretic *lcm*. For example, consider P_5 and C_4. Both have 4 edges. *lcm*(4, 4) = 4 but *lcm*(P5,C4) = 8.



The cartesian product of two graphs *G* and *H* is a graph with vertex set *V (G) x V (H)* for which {(x, u), (y, v)} is an edge if x = y and {u, v} ∈ E(H), or {x, y} ∈ E(G) and u = v. In this paper we have found least common multiple of P4 and $C_m X C_n$ where $C_m X C_n$ is the cartesian product of C_m and C_n. Subsequently we have also found the necessary and sufficient condition for the P_4 decomposability of $C_m X C_n$. For the following discussion, (v_1, v_2, v_3, v_4) denotes a P_4, a path on the vertices v_1, v_2, v_3, v_4. This is a beginning in this area, since least common multiple of two graphs, where at least one of them is a product graph has not been investigated so far.

2. Materials and Methods

For the research we have done to obtain the results mentioned in this paper, we referred the paper by G. Chartrand et.al [2]. For basic results and notations we referred the book by R. Balakrishnan and K. Ranganathan [1].

The proof technique used here is constructive(i.e construction of a least common multiple graph) and have written arguments to justify our claims rigorously like to prove most of the Mathematical assertions.

Results and Discussion

Theorem 1. $lcm(P_4, C_m \times C_n) = \begin{cases} 2mn & if\ mn \equiv 0(\mod 3) \\ 6mn & otherwise \end{cases}$
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Proof. lcm*(P4, C_mxC_n)* is the number of edges in a graph of least size that is both P_4-decomposable and C_m x C_n-decomposable. We consider various cases for m and n in modulo 3 and will construct in each case a graph of least size that is both P_4-decomposable and C_m x C_n-decomposable.

Case 1: m = 3

e(C_3 X C_n) = 6n, which is a multiple of 3. Hence, lcm(3,6n) = 6n. A P_4- decomposition of C_3 X C_n is given by the following copies of P_4:

$$(v_{1,s}, v_{1,s-1}, v_{3,s-1}, v_{3,s}), \quad 2 \le s \le n.$$
$$(v_{1,s}, v_{2,s}, v_{2,s-1}, v_{3,s-1}), \quad 2 \le s \le n.$$
$$(v_{1,1}, v_{2,1}, v_{2,n}, v_{3,n}).$$
$$(v_{1,1}, v_{1,n}, v_{3,n}, v_{3,1}).$$

Thus, $lcm(P_4, C_3 \times C_n) = 6n.$

Figure 1: A P_4-decomposition of C_3 X C_3.

Case 2: $m = 3k, n = 3l$, where $k \ge 2, l \ge 2$.

$e(C_{3k} \times C_{3l}) = 18kl \equiv 0(mod\ 3)$. Hence, $lcm(3, 18kl) = 18kl$. A P_4- decomposition of $C_{3k} \times C_{3l}$ is given by the following copies of P_4:

$$(v_{r,s}, v_{r,s+1}, v_{r,s+2}, v_{r,s+3}), \quad 1 \le r \le 3k,\ s = 3j-2 \text{ where } 1 \le j \le l-1.$$
$$(v_{r,s}, v_{r+1,s}, v_{r+2,s}, v_{r+3,s}), \quad r = 3j-2 \text{ where } 1 \le j \le k-1,\ 1 \le s \le 3l.$$
$$(v_{r,1}, v_{r,3l}, v_{r,3l-1}, v_{r,3l-2}), \quad 1 \le r \le 3k.$$
$$(v_{1,s}, v_{3k,s}, v_{3k-1,s}, v_{3k-2,s}), \quad 1 \le s \le 3l.$$

Thus, $lcm(P_4, C_{3k} \times C_{3l}) = 18kl.$

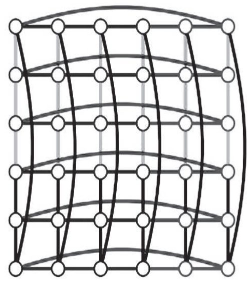

Figure 2: A P_4- decomposition of $C_6 \times C_6$.

Case 3: $m = 3k+1, n = 3l$, where $l \geq 2$.
$e(C_{3k+1} \times C_{3l}) = 6(3k+1)l \equiv 0(\mod 3)$. Hence, $lcm(3, 6(3k+1)l) = 6(3k+1)l$. A P_4-decomposition of $C_{3k+1} \times C_{3l}$ is given by the following copies of P_4:

$$\begin{array}{ll} (v_{r,s}, v_{r,s+1}, v_{r,s+2}, v_{r,s+3}), & 2 \leq r \leq 3k,\ s = 3j-2 \text{ where } 1 \leq j \leq l-1\,. \\ (v_{r,s}, v_{r+1,s}, v_{r+2,s}, v_{r+3,s}), & r = 3j-2 \text{ where } 1 \leq j \leq k,\ 1 \leq s \leq 3l. \\ (v_{1,s}, v_{1,s-1}, v_{3k+1,s-1}, v_{3k+1,s}), & 2 \leq s \leq 3l. \\ (v_{r,1}, v_{r,3l}, v_{r,3l-1}, v_{r,3l-2}), & 2 \leq r \leq 3k. \\ (v_{1,1}, v_{1,3l}, v_{3k+1,3l}, v_{3k+1,1}). & \end{array}$$

Thus, $lcm(P_4, C_{3k+1} \times C_{3l}) = 6(3k+1)l$.

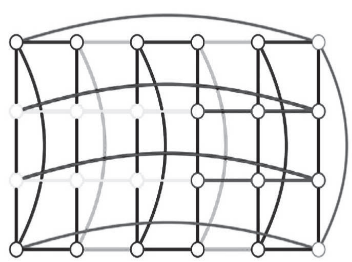

Figure 3: A P_4- decomposition of $C_4 \times C_6$

Case 4: $m = 3k+2, n = 3l$, where $l \geq 2$.
$e(C_{3k+2} \times C_{3l}) = 6(3k+2)l \equiv 0(\mod 3)$. Hence, $lcm(3, 6(3k+2)l) = 6(3k+2)l$. A P_4- decomposition of $C_{3k+2} \times C_{3l}$ is given by the following copies of P_4:

$$\begin{array}{ll} (v_{r,s}, v_{r,s+1}, v_{r,s+2}, v_{r,s+3}), & 2 \leq r \leq 3k+2,\ s = 3j-2 \text{ where } 1 \leq j \leq l-1\,. \\ (v_{r,s}, v_{r+1,s}, v_{r+2,s}, v_{r+3,s}), & r = 3j-2 \text{ where } 1 \leq j \leq k,\ 1 \leq s \leq 3l. \\ (v_{1,s}, v_{1,s+1}, v_{3k+2,s+1}, v_{3k+1,s+1}), & 1 \leq s \leq 3l-1. \\ (v_{r,1}, v_{r,3l}, v_{r,3l-1}, v_{r,3l-2}), & 2 \leq r \leq 3k+2. \\ (v_{3k+1,1}, v_{3k+2,1}, v_{1,1}, v_{1,3l}). & \end{array}$$

Thus, $lcm(P_4, C_{3k+2} \times C_{3l}) = 6(3k+2)l$.

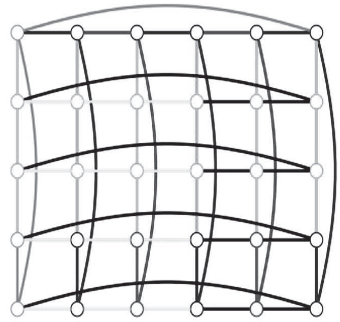

Figure 4: A P_4- decomposition of $C_5 \times C_6$

Case 5: $m = 3k+1, n = 3l+1$.
$e(C_{3k+1} \times C_{3l+1}) = 2(3k+1)(3l+1) \not\equiv 0(mod3)$ and so $lcm(3, 2(3k+1)(3l+1)) = 6(3k+1)(3l+1)$. Let G^i for $i = 1,2,3$ denote three copies of $C_{3k+1} \times C_{3l+1}$. Let $v^i_{r,s}$ denote the $(r,s)^{th}$ vertex of G^i. Identify the vertex $v^1_{3k+1,3l+1}$ with the vertex $v^2_{1,1}$ and the vertex $v^2_{1,3l+1}$ with $v^3_{1,1}$. Let G denote the resulting graph. Clearly, G is $C_{3k+1} \times C_{3l+1}$-decomposable. A P_4 decomposition of G is given by the following copies of P_4:

$(v^i_{r,s}, v^i_{r,s+1}, v^i_{r,s+2}, v^i_{r,s+3})$,	$2 \le r \le 3k$, $s = 3j-2$ where $1 \le j \le l$, $i = 1,2,3$.
$(v^i_{r,s}, v^i_{r+1,s}, v^i_{r+2,s}, v^i_{r+3,s})$,	$r = 3j-2$ where $1 \le j \le k$, $2 \le s \le 3l$, $i = 1,2,3$.
$(v^i_{1,s}, v^i_{1,s-1}, v^i_{3k+1,s-1}, v^i_{3k+1,s})$,	$2 \le s \le 3l+1$, $i = 1,2,3$.
$(v^i_{r,1}, v^i_{r+1,1}, v^i_{r+1,3l+1}, v^i_{r,3l+1})$,	$1 \le r \le 3k$, $i = 1,2,3$.
$(v^1_{1,1}, v^1_{1,3l+1}, v^2_{1,1}, v^2_{1,3l+1})$.	
$(v^2_{3k+1,3l+1}, v^2_{1,3l+1}, v^3_{1,3l+1}, v^3_{3k+1,3l+1})$.	

Thus, $lcm(P_4, C_{3k+1} \times C_{3l+1}) = 6(3k+1)(3l+1)$.

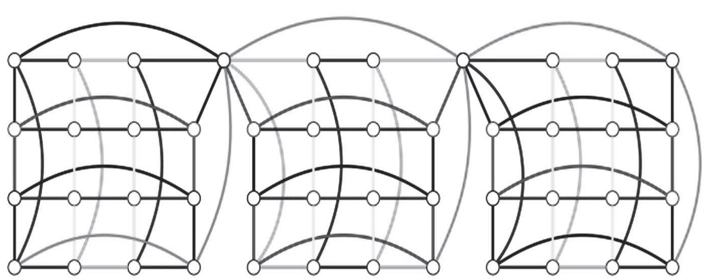

Figure 5: A P_4- decomposition of $C_4 \times C_4$

Case 6: $m = 3k+2, n = 3l+1$.
$e(C_{3k+2} \times C_{3l+1}) = 2(3k+2)(3l+1) \not\equiv 0(mod\ 3)$. Hence, $lcm(3, 2(3k+2)(3l+1)) = 6(3k+2)(3l+1)$. Let G^i for $i = 1,2,3$ denote a $C_{3k+2} \times C_{3l+1}$ graph. Let $v^i_{r,s}$ denote the $(r,s)^{th}$ vertex of G^i. Identify the vertex $v^1_{1,3l+1}$ with the vertex $v^2_{1,1}$ and the vertex $v^2_{1,3l+1}$ with $v^3_{1,1}$. Let G denote the resulting graph. Clearly, G is $C_{3k+2} \times C_{3l+1}$-decomposable. A P_4 decomposition of G is given by the following copies of P_4:

$(v^i_{r,s}, v^i_{r,s+1}, v^i_{r,s+2}, v^i_{r,s+3})$, $2 \le r \le 3k+1$, $s = 3j-2$, where $1 \le j \le l$, $i = 1,2,3$.
$(v^i_{r,s}, v^i_{r+1,s}, v^i_{r+2,s}, v^i_{r+3,s})$, $r = 3j-1$, where $1 \le j \le k$, $2 \le s \le 3l$, $i = 1,2,3$.
$(v^i_{2,s}, v^i_{1,s}, v^i_{3k+2,s}, v^i_{3k+2,s+1})$, $2 \le s \le 3l$, $i = 1,2,3$.
$(v^i_{r,1}, v^i_{r+1,1}, v^i_{r+1,3l+1}, v^i_{r,3l+1})$, $1 \le r \le 3k+1$, $i = 1,2,3$.
$(v^i_{1,2}, v^i_{1,1}, v^i_{3k+2,1}, v^i_{3k+2,2})$, $i = 1,2,3$
$(v^i_{1,3l-1}, v^i_{1,3l}, v^i_{1,3l+1}, v^i_{3k+2,3l+1})$, $i = 1,2,3$
$(v^1_{1,1}, v^1_{1,3l+2}, v^3_{1,1}, v^3_{1,3l+2})$.

Thus, $lcm(P_4, C_{3k+2} \times C_{3l+1}) = 6(3k+2)(3l+1)$.

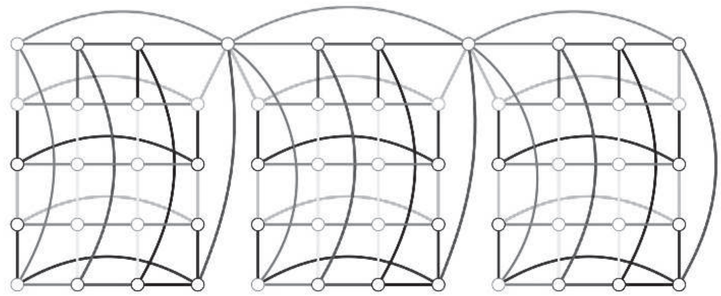

Figure 6: A P_4- decomposition of $C_5 \times C_4$

Case 7: $m = 3k+2, n = 3l+2$. $e(C_{3k+2} \times C_{3l+2}) = 2(3k+2)(3l+2) \not\equiv (0 mod\ 3)$. Hence, $lcm(3, 2(3k+2)(3l+2)) = 6(3k+2)(3l+2)$. Let G^i for $i = 1,2,3$ denote a copy of $C_{3k+2} \times C_{3l+2}$. Let $v^i_{r,s}$ denote the $(r,s)^{th}$ vertex of G^i. Identify the vertex $v^1_{3k+2,1}$ with the vertex $v^2_{1,1}$ and the vertex $v^2_{1,3l+2}$ with $v^3_{3k+2,1}$. Let G denote the resulting graph. Clearly, G is $C_{3k+2} \times C_{3l+2}$-decomposable. A P_4- decomposition of G is given by the following copies of P_4:

$(v^i_{r,s}, v^i_{r,s+1}, v^i_{r,s+2}, v^i_{r,s+3})$, $2 \le r \le 3k+1$, $s = 3j-1$ where $1 \le j \le l$, $i = 1,2,3$.
$(v^i_{r,s}, v^i_{r+1,s}, v^i_{r+2,s}, v^i_{r+3,s})$, $r = 3j-1$ where $1 \le j \le k$, $1 \le s \le 3l+1$, $i = 1,2,3$.
$(v^i_{2,s}, v^i_{1,s}, v^i_{3k+2,s}, v^i_{3k+2,s+1})$, $2 \le s \le 3l+1$, $i = 1,2,3$.
$(v^i_{r,2}, v^i_{r,1}, v^i_{r,3l+2}, v^i_{r-1,3l+2})$, $2 \le r \le 3k+2$, $i = 1,2,3$.
$(v^i_{2,1}, v^i_{1,1}, v^i_{1,2}, v^i_{1,3})$. $i = 1,2,3$
$(v^i_{1,3l}, v^i_{1,3l+1}, v^i_{1,3l+2}, v^i_{3k+2,3l+2})$. $i = 1,2,3$
$(v^1_{1,3l+2}, v^1_{1,1}, v^2_{1,1}, v^2_{3k+2,1})$.
$(v^2_{1,1}, v^2_{1,3l+2}, v^3_{1,1}, v^3_{1,3l+2})$.

Thus, $lcm(P_4, C_{3k+2} \times C_{3l+2}) = 6(3k+2)(3l+2)$.

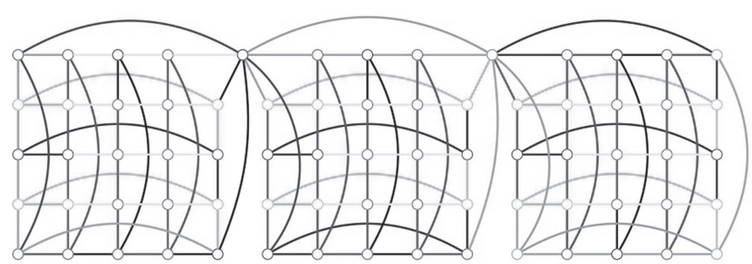

Figure 7: A P_4- decomposition of $C_5 \times C_5$

Theorem 2. $C_m \times C_n$ is P_4- decomposable if and only if $e(C_m \times C_n) = 0$ mod 3.

Proof. This result is a consequence of Theorem 1.

The P_4- decomposability of a graph is an interesting and much discussed open problem in Graph theory. The above theorem gives characterisation for the P_4-decomposability of cartesian product of cycles.

Conclusion

In this paper we have proved the following two theorems. This is a new beginning in the area of common multiple of graphs, since least common multiple of pairs of graphs where at least one of them is a general product graph has not been found in literature. This also opens up a lot of problems for further research.

1. $lcm(P_4, C_m \times C_n) = \begin{cases} 2mn & \text{if } mn \equiv 0(\mod 3) \\ 6mn & \text{otherwise} \end{cases}$

2. $C_m \times C_n$ is P_4- decomposable if and only if $e(C_m \times C_n) \equiv 0 \mod 3$.

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