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GROUP U(N) AND CAYLEY GRAPHS

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Abstract

We show that the group isomorphism $U(st) = U_s(st) \times U_t(st) \approx U(t) \otimes U(s)$ [3] is preserved as graph isomorphism by the Cayley graph of U(st) and Cartesian product of Cayley graphs of $U_s(st)$, $U_t(st)$. And note that the general case of such a graph isomorphism is not true for Cayley graphs of finite abelian groups and cartesian product of Cayley graph of its subgroups, though we have general results in group theory.

Keywords **:** *Group of Units, Cayley Graphs, Cartesian Product*

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1. Introduction

Group of units, U(n), is investigated by Gauss in his classic book Disquisitiones Arithmaticae but the results we are discussing here are due to Joseph Gallian [**3**]. The elements of the multiplicative group integer modulo n, U(n) are the positive integers less than or equal to n and relatively prime to n under multiplication modulo n [**5, 6**]. The order of U(n) is ϕ(n), the Euler phi fuction of n.

EXAMPLE 1.1. U(8) = {1,3,5,7}
|U(8)| = 4
$\phi = 8(1\text{-}\frac{1}{2}) = \frac{8}{2} = 4$.

DEFINITION 1.1. If k|n, let U_k(n)={ x ∈ U(n): x ≡ 1 mod k }.Then U_k(n) is a subgroup of U(n).

EXAMPLE 1.2. Consider U(10) = { 1,3,7,9}
$U_5(10)$ = { x ∈ U(10): x ≡ 1 mod 5} = {1}.
$U_2(10)$ = {x ∈ U(10): x ≡ 1 mod 2} = {1,3,7,9}.

1

THEOREM 1.1. *Suppose s and t are relatively prime. Then U(st) is the internal direct product of $U_s(st)$ and $U_t(st)$, and U(st) is isomorphic to the external direct product of U(s) and U(t). Moreover, $U_s(st) \approx U(t)$ and $U_t(st) \approx U(s)$, so $U(st) = U_s(st) \times U_t(st) \approx U(t) \otimes U(s)$* [**3**].

2. Cayley Graphs and Cartesian Product

A Cayley graph is a graphical representation of a group named after Arthur Cayley who introduced the notion [**7**].

DEFINITION 2.1. Suppose that G is a group and S is a generating set of G. The Cayley graph Γ = Γ(G, S) is a colored directed graph constructed as follows:

(1) Each element g of G is assigned a vertex: the vertex set V(Γ) of Γ is identified with G.
(2) Each generator s of S is assigned a color c_s.
(3) For any g ∈ G and s ∈ S, the vertices corresponding to the elements g and gs are joined by a directed edge of colour c_s. Thus the edge set E(Γ) consists of pairs of the form (g, gs), with s ∈ S providing the color.

EXAMPLE 2.1. Consider the group $\langle \mathbb{Z}_6, +_6 \rangle$. {2, 3} be one of the generating set of the set $\mathbb{Z}_6$. Then we consider the Cayley graph of the group $\langle \mathbb{Z}_6, +_6 \rangle$ with respect to the generating set {2, 3}(where $\mathbb{Z}_6$ = {0, 1, 2, 3, 4, 5}). Then we get the Cayley graph as Figure 1 .

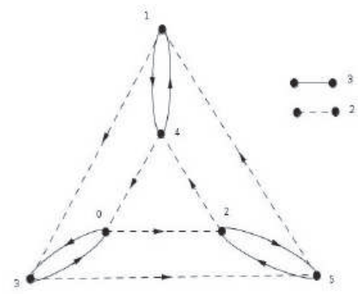

FIGURE 1. Cayley graph of $\mathbb{Z}_6$

2.1. Cartesian Product. The cartesian product of graphs introduced by Gert Sabidussi [**8**] is one of among the major graph products studied in various dimensions.
Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we define the Cartesian product of graphs [**2**] G_1 and G_2 as follows.

DEFINITION 2.2. The Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graphs G = (V, E)(where G = $G_1 \square G_2$), and the V and E defined as follows:

(1) V = $\{(x_1, x_2) : x_1 \in V_1, x_2 \in V_2\}$;
(2) u = $(x_1, x_2) \in$ V and v = $(y_1, y_2) \in$ V, (u, v) is and edge in E, if and only if (x_1, y_1) is an edge in E_1 and $x_2 = y_2$, or (x_2, y_2) is an edge in E_2 and $x_1 = y_1$.

The edge (u, v) is called a G_1 - edge if $(x_1, y_1) \in E_1$, and it is called a G_2 - edge if $(x_2, y_2) \in E_2$. Also x_1 is called the G_1 - component of u and x_2 is called it's G_2 - component.

EXAMPLE 2.2. The Cartesian product of graphs of K_2 and P_4 is given in Figure 2.

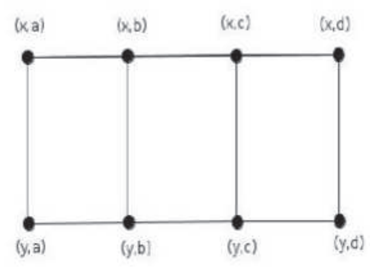

FIGURE 2. Graph of $K_2 \square P_4$

The next theorem is from [**9**]

THEOREM 2.1. *Let* $\Gamma = Cay(G, S)$ *and* $\Gamma' = Cay(G', S')$ *be two Cayley digraphs, then the Cartesian product* $\Gamma \square \Gamma'$ *is the Cayley digraph of the direct product* $G \times G'$ *and the generating subset of* $G{\times}G'$ *is* $(S \times I) \cup (I \times S')$*, denoted by* $\Theta = Cay(G \times G', (S \times I) \cup (I \times S'))$.

This theorem gives , Cartesian product of two Cayley graphs is again a Cayley graph. And we know group isomorphism does not imply corresponding graph isomorphism [**4**]. But in this case we will prove this implication.

3. Main Results

Now we consider the Cayley graphs of U(st) and the Cartesian product of its subgraphs $U_s(st) \times U_t(st)$(where gcd(s, t) = 1)
Let the Cayley graph of U(st) is with the generating set S and Cayley graph of $U_s(st) \times U_t(st)$ is with the generating set $(S'_1 \times I) \cup (I \times S'_2)$ where S'_1, S'_2 are generating sets of $U_s(st), U_t(st)$ respectively.

LEMMA 3.1. *Let* $\psi : S \to (S'_1 \times I) \cup (I \times S'_2)$*, for* $a \in S$

$$\psi(a) = \begin{cases} (a,1), & \text{if } a \in S'_1 \\ (1,a), & \text{if } a \in S'_2 \end{cases}$$

Then ψ *is a bijection.*

PROOF. Now Consider, Case-1: a, b $\in S'_1$, a $\neq$ b. $\Rightarrow$ (a, 1) $\neq$ (b, 1) $\Rightarrow \psi$(a) $\neq \psi$(b).
Case-2 a, b $\in S'_2$, a $\neq$ b $\Rightarrow$ (1, a) $\neq$ (1, b) $\Rightarrow \psi$(a) $\neq \psi$(b) .
Case-3 a $\in S'_1$, b $\in S'_2$ $\Rightarrow$ a $\neq$ b $\Rightarrow$ (a, 1) $\neq$ (1, b) $\Rightarrow \psi$(a) $\neq \psi$(b) .
Hence ψ is one to one.
Again, For any a $\in S'_1$, (a, 1) $\in S'_1 \times$ I. Then (a, 1) = ψ(a)(by definition).
For any b $\in S'_2$, (1, b) $\in$ I $\times S'_2$. Then (1, b) = ψ(b)(by definition).
Hence ψ is bijective. □

Next we are going to the main result.

THEOREM 3.1. *Cayley graph of U(st) and* $U_s(st) \times U_t(st)$ *are isomorphic.*

PROOF. We have U(st) $\approx U_s(st) \times U_t(st)$. [**3**]
Consider their corresponding Cayley graphs. Suppose G be the Cayley graph of U(st) and H be the Cayley graph of $U_s(st) \times U_t(st)$. We have to show that G and H are isomorphic.
Clearly |U(st)| = | $U_s(st) \times U_t(st)$ |. Hence graph G and H have same number of vertices and also there is a bijection(say η) between the vertex set of Cayley graph of U(st) and $U_s(st) \times U_t(st)$.

Since U(st) and $U_s(st) \times U_t(st)$ are finitely generated, the generating sets have finite number of elements.Let S be the generating set of U(st), S_1' be generating set of $U_s(st)$ and S_2' be generating set of $U_t(st)$. So we have $(S_1' \times I) \cup (I \times S_2')$ be the generating set of $U_s(st) \times U_t(st)$ (See Theorem2.1).
Then consider the Cayley graphs G and H, with respect to their generating sets.Then in graph G, let v be the vertex of G, then v is adjacent to vs (by definition of Cayley graph, s $\in$ S and v $\in$ V(G)).
That is (v, vs) is an edge in G.
Next consider (a, b) $\in$ V(H), where a $\in U_s(st)$ and b $\in U_t(st)$.
Then (a, b) is adjacent to (a, b)$(S_1' \times I)$ and (a, b) is adjacent to (a, b)$(I \times S_2')$.
So we have to find out a bijection between the edges of G and H.
Let ϕ : E(G) $\rightarrow$ E(H). Consider v $\in$ V(G) and (a, b) $\in$ V(H).
Then ϕ((v, vs)) = η(v)ψ(s) = (a, b)ψ(s).
Where ψ is a bijection and η is a bijection from V(G) to V(H). Hence ϕ is a bijection(Composition of 2 bijections is again a bijection).
Hence G $\simeq$ H. □

EXAMPLE 3.1. Let n= 6, U(6) = {1, 5}, Generator {5}
U(6) = $U_3(6) \times U_2(6)$ = {1} × {1, 5}. Hence we get Cayley graph of U(6) and $U_3(6) \times U_2(6)$ are isomorphic.
Now for n=12, U(12) = {1, 5, 7, 11}, Generator is {5, 7}
U(12) = $U_3(12) \times U_4(12)$ = {1, 7} × {1, 5}
That is we get Cayley graph of U(12) and $U_3(12) \times U_4(12)$ are isomorphic.

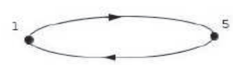

FIGURE 3. Graph of U(6)

FIGURE 4. Graph of $U_3(6) \times U_2(6)$

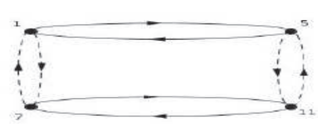

FIGURE 5. Graph of $U(12)$

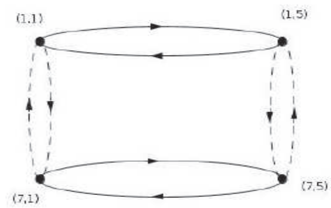

FIGURE 6. Graph of $U_3(12) \times U_4(12)$

4. Conclusion

All the above graphs and its products verifies Cayley graph of U(st) is isomorphic to Cartesian product of Cayley graphs of U_s(st) and U_t(st)(where gcd(s, t) =1).

For settling the general problem for finitely generated abelian groups using fundamental theorem of finitely generated abelian groups[1] we need to have an equivalent version of the result $Z_{mn} \approx Z_m \times Z_n$ if and only if gcd(m, n) = 1 to be 6 GROUP U(N) AND CAYLEY GRAPHS true for Cayley graph of Z_{mn} and Cartesian product Cayley graphs of Z_m, Z_n. But the isomorphism is not preserved for Cayley graphs so we cannot have the general result.

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