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A STUDY ON *m*-STEP GRAPHS AND THEIR ADVANCED APPLICATION IN GRAPH THEORY

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Abstract

In this paper, we analyse and study the m-step graphs, which is an important theoretical construct closely related to the topical research on neighbourhood graphs. We describe the m-step graphs for particular graph classes and critically examine whether some of the graph properties like completeness,hamiltonicity and connectivity are preserved by the m-step function. In many scientic disciplines, trees are the most frequently occurring graphs. In this context, we discuss the structural properties of 2- step graph of trees. Many advanced and recent developments as well as applications are also observed upon.

Keywords **:** *m-Step Graphs, m-Step Competition Graphs, Competition Graphs.*

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1. Introduction and Basic Concepts

Joel E Cohen introduced the notion of competition graph in connection with a problem in ecology in 1968. He brought out this interesting interconnection between mathematics and ecology through his paper [1]. Cohen corresponded a digraph D to a food web. He even associated every vertex of the digraph ,a species in the food web and an arc directed from x to a to the fact that species x preys on species a. If two species x and y have a common prey,they will compete for the prey a. Thus J.E Cohen defined a graph which represents the relations of competition among different species which indeed showcased the food web in nature. The notion of m step competition graph was introduced in 2000. In the m-step competition graph of a digraph, an edge exists between two vertices if both have directed m-step walks to a third vertex in the digraph. Moreover m-step graphs are defined for undirected graphs.

Definition 1.1. If $G = (V, E)$ and $m \in \mathbb{N}$. The (open) m neighbourhood of $x \in V$ is given by $p_m(x : G) = \{y \in V \mid \exists\ xy \text{ path of length } m \text{ in } G\}$. If the context to G is clear, we write $p_m(x)$ for short. p_m is symmetric for undirected graphs: $y \in p_m(x) \Leftrightarrow x \in p_m(y)$.

Definition 1.2. If $G = (V, E)$ is the graph, then its m-step graph $N_m(G) = (V, E_m)$ where E_m is given by $E_m = \{xy \mid y \in p_m(x)\}$.

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The trivial cases of definition 1.2 are the following

- The 1-step graph of G is G itself, because paths of length of 1 in G are given exactly by its edges $E(G)$.
- For $m \geq |V|$, the m-step graphs of G has no edges because, there is no path of length m with $|V|$ vertices.

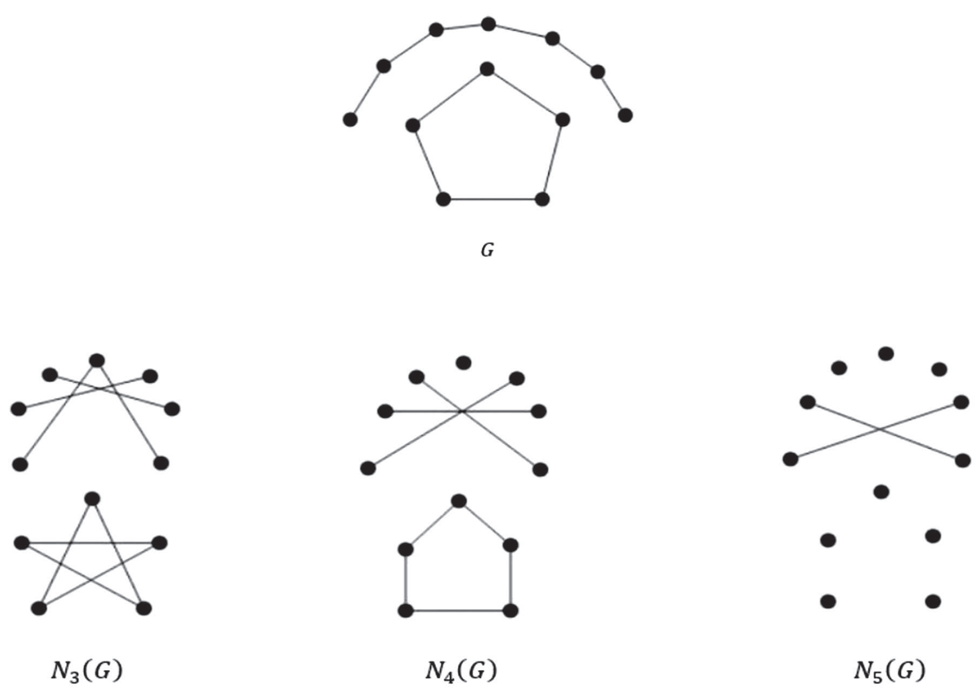

FIGURE 1. The 3−, 4− and 5− step graphs of graph G

Theorem 1.3. *Let G be a simple graph. Then $H \subseteq G$ implies $N_m(H) \subseteq N_m(G)$*

Proof. Let $H \subseteq G$ be an arbitrary subgraph of G. Since $V(N_m(H) = V(H) \subseteq V(G) = N_m(G)$, it follows that $V(N_m(H)) \subseteq V(N_m(G)$. Now let $xy \in E(N_m(H))$ be arbitrarily chosen. Then there exist an $x - y$ path of length m in H. Since H is a subgraph of G, this path will exist in G also. *i.e.* $y \in p_m(x : G)$. Therefore $xy \in E(N_m(G)$. □

2. m- Step Graph of Particular Graph Classes

The following theorems give the general form of m-step graphs of paths, cycles, complete graphs and wheels

2.1. m **step graph of paths and cycles.**

Theorem 2.1. [2] *Let $d \in [0, m-1]$ with $d \equiv n \pmod m$. The m-step graph $N_m(P_n)$ consists of m paths; d of those have $\lceil \frac{n}{m} \rceil$ vertices and the other paths have $\lfloor \frac{n}{m} \rfloor$ vertices.*

Theorem 2.2. [2] *Let $g = gcd(m, n)$. The m-step graph of C_n consist of g cycles of equal length. i.e,$N_m(C_n) = gC_{\frac{n}{g}}$*

2.2. m**-step graph of Complete Graphs.**

Theorem 2.3. *The m-step graph of $K_n = K_n$ for m = 1,2,...$n-1$*

Proof. Let $V(K_n) = \{v_1, v_2, ...v_n\}$. For $m = 1$, the 1 step graph of K_n is K_n itself. Now consider the case when $m = 2$. For any two arbitrary vertices v_i and v_j, there exists a vertex v_k such that v_k is adjacent to both v_i and v_j. Hence $N_2(K_n) = K_n$. Now suppose that the result holds for $n-2$. i.e, $N_{n-2}(K_n) = K_n$.Then there exists an $(n-2)$- step path between any two vertices of K_n. Consider an arbitrary vertex say v_i. By induction hypothesis, there exists an $(n-2)$- step path from v_i to any other vertex say v_k where $k \neq i$; $k, i = 1, 2...n$. Let that path be P'. Then P' contains $n-1$ vertices. Hence there remains a vertex which is not in P' say v_s. Then $P' \cup v_k \cup v_s$ is an $(n-1)$- step path from v_i to v_s. Hence the result is true for $n-1$. □

2.3. **m step graph of Wheels.**

A Wheel W_n is a graph with one centre vertex connected to each vertex of a cycle on 'n' vertices.

Theorem 2.4. [2] *The m-step graph of a Wheel W_n is the complete graph K_{n+1}*

Remark 2.5. The m-step graph of a complete graph is always complete by Theorem 2.3, however we could clearly infer that there are not merely complete graphs which give their m- step graph to be complete by Theorem 2.4 since the m-step graph of Wheel is a complete graph.

3. Conectivity and Hamiltonicity

Theorem 3.1. [2] *If $N_m(G)$ is connected then G is connected.*

Proof. Let x, $y \in V(G)$ be arbitrary vertices. Since $N_m(G)$ is connected, there is path P from x to y in $N_m(G)$. An edge of this path is induced by a path of length m in G. Thus, there is a walk from x to y in G and hence G is connected. □

Remark 3.2. Converse of the above theorem does not hold in general. We will give a counter example.

FIGURE 2. A connected graph G with its 3-step graph being disconnected

Theorem 3.3. [2] *Let $G = (V, E)$ be a graph with minimum degree $\delta(G)$.And $m \in N$ with $2 \leq m \leq \delta(G)$. Then we obtain for the m-step graph $N_m(G)$, $\delta(N_m(G)) \geq \delta(G) - 1$*

Theorem 3.4. [2] *Let $G = (V, E)$ be a simple graph, $n = |V| \geq 3$ and $m < n$. If $\delta(G)$ $\geq \frac{n}{2} + 1$, then $N_m(G)$ is Hamiltonian.*

Proof. From $\delta(G) \geq \frac{n}{2} + 1$ and Theorem 3.3, it follows that $\delta(N_m(G)) \geq \frac{n}{2}$ which implies a Hamiltonian cycle according to Dirac.

□

4. STRUCTURAL PROPERTIES OF 2-STEP GRAPH OF TREES

The following theorems give some of the structural properties of the 2-Step graph of trees.

Theorem 4.1. [3] *If T is a tree, then $N_2(T)$ is diamond free.*

Theorem 4.2. [3] *If H is a connected subgraph of a tree T, then $N_2(H)$ is an induced subgraph of $N_2(T)$*

Proof. Let $x, y \in V(H)$. It is clear that if $xy \in E(N_2(H))$, then $xy \in E((N_2)(T))$. Now suppose that $xy \notin E(N_2(H))$. There exist no $v \in V(H)$ such that $xv, yv \in E(H)$. Suppose that $T - H$ contains a vertex v such that $xv, yv \in E(T)$. Then there would be in addition to the path from x to y in H, a path from x to y using v, but this is impossible since T is a tree. Thus $xy \notin E(N_2(T))$.

Theorem 4.3. *Let T be a tree and H a component of $N_2(T)$. Then if $u, v \in V(H)$, $d_T(u,v)$ is even.*

Proof. Since u and v lie on the same component of $N_2(H)$,then by definition of $N_2(H)$, there is a path of even length from u to v in T. But since T is a tree,this is the unique $u-v$ path in T and the result follows immediately. □

Advanced research and developments in this area have been occuring related to the competition graphs. Next section describes Competition Graphs.

5. Competition Graphs

The competition Graph $G = C(D)$ corresponding to the directed graph D is the simple graph defined in the following way:
$V(G) = V(D)$ and $E(G) = \{(u,v) \mid \text{there exists } x \in V(D) \text{ for which } \{(u,x),(v,x)\} \subset E(D)\}$.

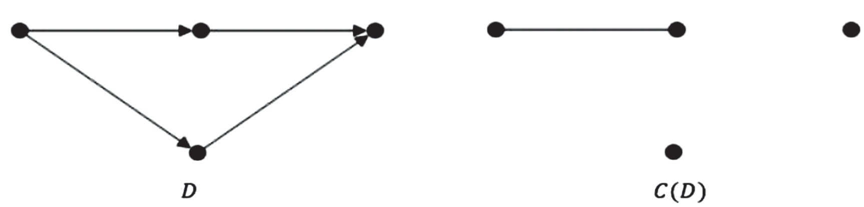

Figure 3. A digraph D with its competition graph $C(D)$

While discussing competition graphs, we often refer to vertices as predators and prey. Vertex x preys on vertex y if $(x,y) \in E(D)$. Moreover, there are a few assumptions made about digraph D because of this application. First, digraphs must be loopless because animals do not prey upon themselves. Secondly, all digraphs D must be acyclic because cycles do not occur naturally in the wilderness. In this way, natural food webs were beautifully presented using competiton graphs. Moreover, another possible use for competition graphs is economic graphs. In this case, the digraph D model asset relationships in economic systems. Because of this, we must change the definition of the digraph slightly in this model.

Remark 5.1. There are different variants of competition graphs namely; many step competition graphs, same step competition graphs, any step competition graphs. Recent researches regarding competition graphs has occurred in fuzzy competition graphs.The species may be strong or weak and also the prey could be poisonous or harmful and hence there is vagueness in relationships between prey and predators. It is in this context that the notion of fuzzy competition graphs have come into play. It helps in studying the soil ecosystem.

6. Research Gap and More Applications

Competition graphs are becoming increasingly significant as they can apply to many areas in which there occurs competition between entities. One conjecture says that even paths are m step graphs and odd paths are not. A further classification of same step competition graphs is yet another topic for future research. In addition to ecology, the various applications of competition graphs even include those to channel assignments, coding, modeling of complex economic and energy systems.

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References

[1] J E Cohen, Interval graphs and food webs : A finding and a problem, *Rand Corporation Document 17696-PR, Santa Monica,CA*, (1968)

[2] Rene Schon Felder, Structural Properties of m –Step Graphs, *University of Lubeck, Institute of Mathematics,Research Report*, (2009)

[3] J Richard Lundgren,Craig W.Rasmussen, Two step graph of trees,*Discrete Applied Mathematics 119 , 123 – 139*, (1993)

[4] Wei Ho , The m-step,same-step,and any-step competition graphs,*Princeton University, Department of Mathematics, Princeton, NJ 08544, USA*,(2005)

[5] Han Hyuk Cho,Suh-Ryung-Kim,Yunsun Nam, The m-step competition graph of a digraph, *Discrete Applied Mathematics 105, 115–127* (2000)

[6] R.D. Dutton, R.C. Brigham, A characterization of competition graphs,*Discrete Applied Mathematics, 6 , 315-317* (1983)