ISSN: 2249-7862 (Print) : 2455-877X (Online)
www.aureoleonline.in

Power Domination in Some Special Graphs and Derived Graphs

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Abstract
A vertex subset S of a graph G is said to be a dominating set of G if all vertices in G are either in S or adjacent to at least one vertex in S. The minimum cardinality of such a set is the domination number of the graph and is denoted as $\gamma(G)$. A vertex subset S of graph G is said to be a power-dominating set if every vertex and every edge in the system is monitored by S followed by domination and propagation. The power domination number $\gamma_P(G)$ of a graph G is the minimum cardinality of power dominating set G. In this study, we discuss the power domination in derived graphs such as shadow and glue graphs. Also, we discuss this concept in some special graphs.

Keywords: domination number; dominating set; power domination number; degree; universal vertex.

Mathematics Subject Classification 2020: 05C69; 05C76.

Introduction

The graphs discussed in this paper are simple, connected and undirected. For the basic graph theoretic terminologies we refer [2, 10]. A vertex v in a graph G is said to be a universal vertex [3] if it is adjacent to all other vertices in G. The number of edges incident to a vertex v is its degree [10] and is denoted as $deg(v)$. A vertex with degree one is the pendant vertex. The maximum and the minimum degree of a graph G are denoted by Δ and δ respectively. An edge e of a graph is said to be a bridge [2], if its removal disconnects the graph. The eccentricity of a vertex v in a graph G is denoted as $e(v)$ and is defined as the distance between v and the farthest vertex in G. The maximum and the minimum eccentricities in a graph G are the diameter [2], $diam(G)$ and the radius, $rad(G)$ of the graph G.

The glue graph [8] of a graph G is denoted as G_g and is obtained from G by making adjacency between the vertices with same eccentricity. The complement [2] of a graph G is denoted as $\overline{G}$ and it consists same vertex set as that of G, where two vertices u and v are adjacent in $\overline{G}$ iff they are not adjacent in G. The shadow graph [7] of a graph G is denoted as $D_2(G)$ and is obtained by taking two copies of G such that there is an adjacency between each vertex in a copy of G to the neighbors of the corresponding vertex in another copy of G. The Figure 1 represents P_{8_g} and Figure 2 represents $D_2(P_8)$.

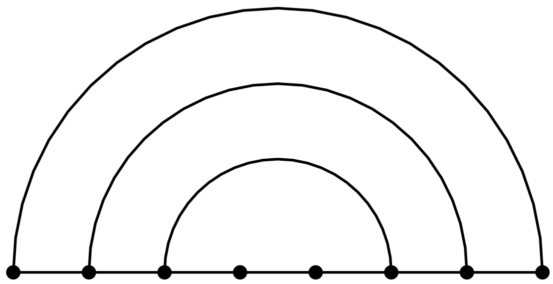

Figure 1: The glue graph of P_8.

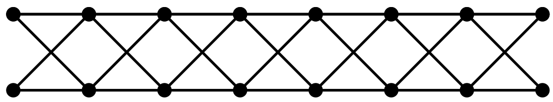

Figure 2: The shadow graph of P_8.

The domination concept in graph theory was developed from the famous five queens problem. A detailed literature about this concept can be seen in [5, 6]. The concept of power domination was raised from the monitoring of an electric power system in which few measurement devices are placed for effective power distribution. The graph theoretical explanation of this problem can be found in [4].

Definition 0.1. *A vertex subset G of a graph G is said to be a dominating set [6], if all vertices in G are either in D or adjacent to at least one vertex in D. The minimum cardinality of such a set is the domination number of G, denoted by $\gamma(G)$.*

An electric power system is graph theoretically represented as follows. The electric nodes in the system are represented by the vertices and the transmission lines between them are represented by the edges between the corresponding vertices. To monitor the entire system we want to place some phase measurement units (PMU) at certain nodes. To reduce the cost, the number of such PMU's must be minimal. For the identification of the minimum number of PMU's we use the domination concept in graph theory. If a PMU is placed in a certain node, then all the nodes that are adjacent to that node and the corresponding transmission lines are observed. In addition to that, we follow the propagation rule also [4]. From this idea the following definition was developed.

Definition 0.2. *A vertex subset S of a graph G is said to be a power dominating set [4] if every vertex and every edge in the system is monitored by the vertex subset D followed by domination and propagation. The minimum cardinality of such a set D is the power domination number, $\gamma_P(G)$.*

Definition 0.3. *A coconut tree graph $CT_{m,n}$ [3] consists of a path P_n such that m pendant vertices are attached to one end of the path.*

Definition 0.4. *An n-barbell graph [3] is obtained by connecting two copies of K_n by a bridge.*

Definition 0.5. *A firecracker graph $F_{n,k}$ [3] is obtained by the connection of n, k-stars by combining one pendant vertex of the star S_n to the pendant vertex of the next star.*

Definition 0.6. *A banana tree graph $B_{n,k}$ [3] is obtained by making adjacency between a pendent vertex from each n copy of star S_k to a new vertex.*

Definition 0.7. *A helm graph H_n [3] is formed by attaching a pendent vertex to each of the vertices except the central vertex in the wheel graph.*

Definition 0.8. *A lollipop graph $L_{m,n}$ [3] is obtained by connecting an end of a path P_n with the complete graph K_m by a bridge.*

Definition 0.9. *A triangular snake graph T_n [9] consists of a path P_n such that any two adjacent vertex x_i and x_{i+1} in P_n are adjacent to a new vertex y_i, where $1 \leq i \leq n-1$.*

Definition 0.10. *A double triangular snake graph DT_n [9] consists of a path P_n such that any two adjacent vertices x_i and x_{i+1} in P_n are adjacent to the new vertices y_i and z_i ($1 \leq i \leq n-1$) in upward and downward directions.*

Definition 0.11. *A diamond snake graph [1] is obtained from a path by connecting vertices to the new vertices u_{i+1} and w_{i+1} for $0 \leq i \leq k-1$. That is every edge of a path $v_0\ v_1.....v_m$ of size m replaced by C_4, where, $d(v_i, v_{i+1}) = 2$; $0 \leq i \leq k-1$.*

Theorem 0.1. *[6] Let G be a graph of order n and diameter d. Then,*
$\lceil \frac{d+1}{3} \rceil \leq \gamma(G) \leq \lceil \frac{n}{2} \rceil$.

Theorem 0.2. *[4] For a cycle C_n and path P_n, $\gamma_P(C_n) = 1 = \gamma_P(P_n)$.*

Remark 0.1. *Let G be a graph with at least one universal vertex, then $\gamma_P(G) = 1$.*

Remark 0.2. *The power domination number is always a lower bound for the domination number of a graph.*

In the following section, we are discussing the power domination in some special graphs and also in derived graphs such as glue graphs and shadow graphs.

Power Domination in a Graph

Theorem 0.3. *For a coconut tree graph $CT_{m,n}$, $\gamma_P(CT_{m,n}) = 1$.*

Proof. Let v be the star's central vertex attached to an endpoint of the path to form $CT_{m,n}$. By domination rule v can dominate all vertices in N(v). Let v_j be the vertex in a path adjacent to v.Then by applying the propagation rule to v and continuing the propagation rule all other vertices in $CT_{m,n}$ get dominated. Hence $\gamma_P(G) = 1$. □

Theorem 0.4. *For an n-barbell graph G, $\gamma_P(G) = 1$.*

Proof. Let $e = xy$ be the bridge in G. Then the vertex x dominates y and all other neighbors of x. Then by propagation rule the remaining vertices in G which are the neighbors of y get dominated. Hence $\gamma_P(G) = 1$. □

Theorem 0.5. *For a firecracker graph $F_{m,n}$, $\gamma_P(F_{m,n}) \leq m$.*

Proof. Let $v_1 \in V(F_{m,n})$, where $deg(v_1) = 1$. Then we can dominate only the corresponding central vertices in S_m. Since there are non-adjacent neighbors for the central vertices, we can't proceed with the propagation rule. Similarly, if we choose all vertices in the path, again it dominates the central vertices, but we can't proceed with the propagation rule. Now consider each central vertices, the set of central vertices dominates all the vertices in $F_{m,n}$. Also the same set forms a power-dominating set. Hence $\gamma_P(G) \leq m$. □

Theorem 0.6. *For a banana tree graph $B_{n,k}$, $\gamma_P(B_{n,k}) \leq n$.*

Proof. Let v be any of the pendant vertex in $B_{n,k}$. Then v can dominate only one of its neighbors, further, we can't proceed, since the propagation rule fails. If we are choosing exactly one pendant vertex from each copy of the star, then we cannot dominate the entire vertices. Now choose the central vertices from each copy. Then by domination and propagation rule, we can dominate all the vertices. Hence $\gamma_P(B_{n,k}) \leq n$. □

Theorem 0.7. *For a helm graph H_n, $\gamma_P(H_n) = 1$.*

Proof. Let v be the central vertex in H_n. Then v dominates all its neighbors. Then applying the propagation rule to each neighbor of v, all pendant vertices in H_n get dominated. Hence $\gamma_P(H_n) = 1$. □

Theorem 0.8. *For a lollipop graph $L_{m,n}$, $\gamma_P(L_{m,n}) = 1$.*

Proof. Consider the graph $L_{m,n}$ in which the path P_n be attached to the vertex u of K_m. Then u dominates all other vertices in K_m and a vertex say v_1 in P_n. Then by applying the propagation rule to v_1, its neighbor v_2, get dominated. Again apply the propagation rule to v_2 and the process repeats. Hence all vertices in $L_{m,n}$ are dominated by the vertex u. Hence $\gamma_P(L_{m,n}) = 1$. □

Theorem 0.9. *For a triangular snake graph T_n, $\gamma_P(T_n) = 1$.*

Proof. Let u be any of the end vertex in the path P_{n+1} of T_n. Then by the rule of domination, these vertex dominate their two neighbors. But these neighbors are adjacent. So we can apply the propogation rule and the process continues and hence all the other vertices get dominated. Therefore, $\gamma_P(T_n) = 1$. □

Theorem 0.10. *For a double triangular snake graph DT_n, $\gamma_P(DT_n) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let P_{n+1} be a path in DT_n, where $V(P_{n+1}) = \{u_1, u_2, \ldots, u_{n+1}\}$. Consider the set $S = \{u_2, u_4, \ldots u_{n+1}\}$. Then S forms a power dominating set of DT_n and hence $\gamma_P(DT_n) \leq \lceil \frac{n}{2} \rceil$. □

Theorem 0.11. *For a diamond snake graph D_n $(n > 1)$, $\gamma_P(D_n) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let $S = \{v \in V(D_n); deg(v) = 4\}$ and there are $n - 1$ such vertices. Consider $S = \{v_1, v_2, \ldots, v_{n-1}\}$. Now consider the following two cases

Case 1: n is even.

The set $\{v_1, v_3, \ldots, v_{n-1}\}$ form a power dominating set of D_n and hence $\gamma_P(D_n) \leq \frac{n}{2}$.
Case 2: n is odd.
The set $\{v_1, v_3, \ldots, v_{n-2}, u_{n+1}\}$, form a power dominating set of D_n, where u_{n+1} is a vertex in D_n with $deg(u_{n+1}) = 2$ and its neighboring vertices are also of degree two. Hence $\gamma_P(D_n) \leq \lceil \frac{n}{2} \rceil$. □

Theorem 0.12. *For any $r-$star graph $G = S_{n_1,n_2,\ldots,n_r}$, $\gamma_P(G) \leq r$.*

Proof. The proof is similar to the proof of Theorem 0.5. □

Theorem 0.13. *For a glue graph of the path P_n, $\gamma_P(P_{n_g}) = 1$.*

Proof. Let v be any of the pendent vertex of path P_n. Then in the glue graph of P_n, v is adjacent to two vertices. Then v dominates these two vertices and applies the propagation rule to these two vertices. Then repeat the process and finally all the remaining vertices are dominated by this vertex v. Hence $\gamma_P(G) = 1$. □

Theorem 0.14. *For a shadow graph of a path P_n $(n > 2)$, $\gamma_P(D_2(P_n)) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. For n is odd, the alternate vertices in a copy of P_n with degree four form the power dominating set of $D_2(P_n)$. For n is even, the alternate vertices in a copy of P_n with degree four and a vertex in $D_2(P_n)$ with degree two form the power dominating set. Hence $\gamma_P(D_2(P_n)) \leq \lfloor \frac{n}{2} \rfloor$. □

Theorem 0.15. *For the complement of a path P_n $(n > 4)$, $\gamma_P(\overline{P}_n) = 1$.*

Proof. Let v_1 be a pendant vertex in P_n which is adjacent to the vertex v_2. Then in $\overline{P}_n$, v_1 is adjacent to all other vertices. Therefore, in $\overline{P}_n$, v_1 dominates all vertices except v_2. Then by propagation rule v_2 also gets dominated. Hence $\gamma_P(\overline{P}_n) = 1$. □

Conclusion

In this work, we studied the power domination in certain special graphs and derived graphs such as glue graphs and shadowgraphs. There are wide areas related to power domination yet to be explored like in derived graphs such as line graphs, edge-subdivision graphs and power of a graph etc. Also, the study concerning different graph operations, and for other domination variants is a better option for further studies. Since this concept is developed from the electric power system, it can also be extended concerning the other domination variants.

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