ISSN: 2249-7862 (Print) : 2455-877X (Online)
www.aureoleonline.in

A Study on Topological Indices in Graphs

Karthika Harish[1], **Aswathy Samuel**[1] and **John Joy Mulloor**[2,*]
[1]Department of Mathematics, BCM College, Kottayam
[2]Department of Mathematics, St. Stephen's College, Uzhavoor
[*]E-mail: johnjoy.mulloor@gmail.com

Abstract
The topological indices are numerical quantities associated with graphs that are invariant under the graph isomorphism. In a connected graph $G = (V, E)$, the Albertson index is defined as $Alb(G) = \sum_{e=uv \in E} |deg(u) - deg(v)|$ and the sigma index is defined as $\sigma(G) = \sum_{e=uv \in E} (deg(u) - deg(v))^2$. In this study, we are discussing the Albertson index and sigma index in some derived graphs and special graphs.

Keywords: Albertson index; sigma index; glue graphs; complement; shadow graphs.

Mathematics Subject Classification 2020: 05C09; 05C92.

Introduction

The graphs discussed in this paper are simple, connected and undirected. We refer to the paper [3, 10] for the basic graph theoretic concepts. For the graph $G = (V, E)$, V denote the vertex set and E is its edge set. The cardinality of V and E is the order and size [10] of the graph respectively. The maximum degree of the graph is denoted as Δ and the minimum degree is denoted as δ [3]. The degree of a vertex v is denoted as $deg(v)$ and is defined as the number of edges incident to the vertex v. A graph G is said to be regular if the degree of each vertex in G is the same. A bridge [10] in a graph G is an edge in G whose removal disconnects G.

The eccentricity [3] of a vertex v in a graph G is denoted by $e(V)$ and is defined as the distance between v and a vertex farthest from v in G. The maximum eccentricity is the diameter [3] $diam(G)$ and the minimum is the radius $rad(G)$ of the graph. A graph G is said to be self-centered if the eccentricity of each vertex in G is the same. In other words, if the diameter and radius of a graph are the same, then the graph is self-centered [9]. A vertex v is said to be a central vertex of G if $e(v) = rad(G)$.

The glue graph G_g [9] of a graph G is obtained by making adjacency between the vertices with the same eccentricity in G. The shadow graph $D_2(G)$ [8] consists of two copies of graph G such that there is an adjacency between the vertex u in G with the neighbors of the corresponding vertex of u in the other copy. The complement of a graph G is denoted as $\overline{G}$ [3] and it consists of the same vertex set that of G provided two vertices in $\overline{G}$ are adjacent if and only if they are non-adjacent in G.

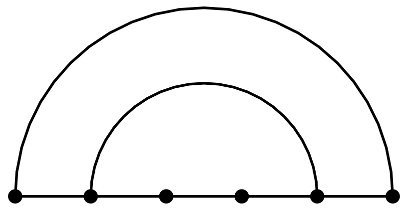

Figure 1: The glue graph of P_6.

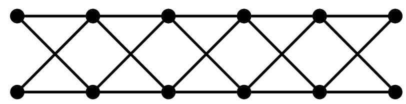

Figure 2: The shadow graph of P_6.

The topological indices are numerical quantities associated with graphs that are invariant under the graph isomorphism. The study on topological indices has applications related to the study on chemical compounds. The different types of topological indices

such as the Wiener index, sigma index, Albertson index and Zagreb index, etc. can be seen in [1, 6, 7]. The Albertson index and the sigma index are introduced based on the degree of the vertices and these indices explain the irregularity of the graph. The Albertson index and the sigma index of a regular graph are zero.

Definition 0.1. *In a connected graph G, the Albertson index [1] is defined as*

$$Alb(G) = \sum_{e=uv\in E} |deg(u) - deg(v)|.$$

Definition 0.2. *In a connected graph G, the sigma index [7] is defined as*

$$\sigma(G) = \sum_{e=uv\in E} (deg(u) - deg(v))^2.$$

Definition 0.3. *For $m, n \in \mathbb{N}$, a Jellyfish graph $J_{m,n}$ [4] consists of the vertex set $V(J_{m,n}) = \{u, v, x, y, x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\}$ and the edge set $E(J_{m,n}) = \{(x, x_i), (x, u)$ $, (x, v), (u, v), (y, u), (y, v), (y, v_j);\ 1 \leq i \leq m\ \text{ and }\ 1 \leq j \leq n\}$.*

Definition 0.4. *A Helm graph H_n [4] is obtained from a wheel graph W_n by attaching a pendant vertex to each vertex in W_n except the central vertex.*

Definition 0.5. *A lollipop graph $L_{m,n}$ [4] consists of P_n and K_m which are connected by a bridge.*

Definition 0.6. *A Diamond snake graph [2] is got from a path by connecting adjacent vertices to 2 new vertices* u_{i+1} *and* w_{i+1} *for* $0 \leq i \leq k-1$*. That is every edge of a path* $v_0v_1....v_m$ *of size* m *is replaced by a cycle* C_4 *and* $d(v_i, v_{i+1}) = 2$*;* $0 \leq i \leq k-1$*.*

Definition 0.7. *An* $n-$*barbell graph [4] consists of two copies of* K_n*, which are connected by a bridge.*

Definition 0.8. *A friendship graph* F_n *[4] is of order* $2n+1$ *and is obtained by using* n *number of* K_3*'s with a common vertex.*

Theorem 0.1. *[1, 7] For any regular graph* G*,* $Alb(G) = 0$ *and* $\sigma(G) = 0$*.*

Theorem 0.2. *[5] The Albertson index of a simple graph is even.*

Remark 0.1. *The Albertson index and the Sigma index of glue graph of a self-centered graph is zero.*

Remark 0.2. *The Albertson index and the Sigma index of shadow graph of a regular graph is zero.*

Albertson Index and Sigma Index of a Graph

In this section, we are discussing the Albertson index and the Sigma index in some special graphs and also concerning the complement, glue graph and shadow graph of some classes of graphs.

Theorem 0.3. *For a jellyfish graph* $J_{m,n}$*,* $Alb(J_{m,n}) = m^2 + n^2 + 3m + 3n - 4$ *and* $\sigma(J_{m,n}) = (m(m+1))^2 + (n(n+1))^2 + (2m-2)^2 + (2n-2)^2$*.*

Proof. In $J_{m,n}$, $xx_1, \ldots, xx_m$ have the vertex degree difference $m+1$. There are m such edges. Similarly for the edges, $yy_1, \ldots, yy_n$, vertex degree difference is $(n+1)$. For the edges yu and yv have the degree difference $n-1$. Also, for the edges xu and xv have the degree difference $m-1$. Hence, the Albertson index becomes $m^2+n^2+3m+3n-4$ and Sigma index becomes $(m(m+1))^2+(n(n+1))^2+(2m-2)^2+(2n-2)^2$ on simplification. □

Theorem 0.4. *For a Helm graph* H_n*,* $Alb(H_n) = (n-1)(n-2)$ *and* $\sigma(H_n) = (n-1)(n^2 - 8n + 20)$*.*

Proof. Let v_c be the central vertex in H_n. Then, $deg(v_c) = n-1$. Let S be the set of pendant vertices in H_n. The degree of all vertices in $V(H_n) - (S \bigcup \{v_c\})$ is three. So we need to consider only the pendant edges and all vertices incident to v_c. The vertex degree difference in each pendant edge is two and the vertex degree difference for each edge incident to v_c is $n-4$. Hence on simplification, we get the result. □

Theorem 0.5. *For a lollipop graph* $L_{m,n}$*,* $Alb(L_{m,n}) = 2n-1$ *and* $\sigma(L_{m,n}) = 2n-1$*.*

Proof. The 2 edges joining the complete graph will give a contribution of its degree difference as 1 and it has a total of $n-1$. The bridge joining the complete graph and the last edge of the path has 1 as its degree difference. Thus, its Albertson index and Sigma index of the graph becomes $2n-1$.

□

Theorem 0.6. *For a diamond snake graph G, $Alb(G) = 8(n-1)$ and $\sigma(G) = 16(n-1)$.*

Proof. The vertex set of G can be partitioned into V_1 and V_2, where $V_1 = \{v \in V(G); deg(v) = 2\}$ and $V_2 = \{v \in V(G); deg(v) = 4\}$. There is no edge between any two vertices in V_2. The vertex degree difference between any two vertices in V_1 is zero. So we need to consider only the edges between a vertex from V_1 and a vertex from V_2, the vertex degree difference of such an edge is two and there is $4(n-1)$ such edges. Therefore, $Alb(G) = 8(n-1)$ and $\sigma(G) = 16(n-1)$.

□

Theorem 0.7. *The Albertson index and the Sigma index of an $n-$barbell graph G are the same.*

Proof. Let xy be the bridge in G. Then the degree of all vertex $v \in V(G) - \{x, y\}$ are the same. So we need to consider only the edges that are incident to x and y. But, $deg(x) = deg(y) = n$ and rest all vertices with degree $n-1$. Hence $Alb(G) = n-1 = \sigma(G)$ □

Theorem 0.8. *For a friendship graph F_n, $Alb(F_n) = 4n(n-1)$ and $\sigma(F_n) = 8n(n-1)^2$*

Proof. Let v be the central vertex in F_n. Choose any two vertices $x, y \in V(F_n) - \{v\}$. Then $deg(x) = deg(y)$. Therefore, to calculate the Albertson index and Sigma index we need to consider only the edges which are incident to v. There are $2n$ such edges. Also, $deg(v) = 2n$ and the degree of the rest of the vertices is two. Hence on simplification, we get the result. □

Theorem 0.9. *For the complement of path P_n $(n > 3)$, $Alb(\overline{P}_n) = 2n-6 = \sigma(\overline{P}_n)$.*

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where v_1 and v_n are the pendant vertices. Then in $\overline{P}_n$, $deg(v_1) = deg(v_n) = n-2$ and $deg(v_j) = n-3$, where $2 \leq j \leq n-1$. So we need to consider only the edges v_1v_j and v_nv_j. Since the vertex degree differences of the rest of the edges are zero. But the vertex degree difference of v_1v_j and v_nv_j are equal to one. Also, total $n-3+n-3 = 2n-6$ such edges exist. Hence $Alb(\overline{P}_n) = 2n-6 = \sigma(\overline{P}_n)$. □

Theorem 0.10. *For a shadow graph of a path P_n $(n > 3)$, $Alb(D_2(P_n)) = 16$ and $\sigma(D_2(P_n)) = 32$.*

Proof. Let $V(D_2(P_n)) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ and consider the edge subset $E_1 = \{v_1v_2, v_1v_2', v_1'v_2', v_1'v_2, v_nv_{n-1}, v_nv_{n-1}', v_n'v_{n-1}', v_n'v_{n-1}\}$. The vertex degree difference of each of the edges in E_1 is two. If $e \in E(D_2(P_n)) - E_1$, then the vertex degree difference is zero. Hence there exists 8 edges in $D_2(P_n)$ with a degree difference is two.

Therefore, $Alb(D_2(P_n)) = 16$ and $\sigma(D_2(P_n)) = 32$.

□

Theorem 0.11. *For a glue graph of a path P_n , the Albertson index and Sigma index are the same.*

Proof. The glue graph of P_2, P_3, P_4 are P_2, K_3 and C_4 respectively. Then from Theorem 0.1, the result follows. Let $n > 4$ and v_1, v_n be the end vertices in P_n. Then in P_{n_g}, $|deg(v_1) - deg(v_2)| = 1$ and $|deg(v_n) - deg(v_{n-1})| = 1$. Now consider the following two cases.

Case 1: n is even.
Let v_i and v_j be the central vertices in P_n. Then in P_{n_g}, $|deg(v_j) - deg(v_i)| = 0$. Suppose v_{i-1} and v_{j+1} are the other neighbors of v_i and v_j respectively. Then $|deg(v_j) - deg(v_{j+1})| = 1$ and $|deg(v_i) - deg(v_{i-1})| = 1$. Also, $|deg(v_2) - deg(v_1)| = 1$ and $|deg(v_n) - deg(v_{n-1})| = 1$. For all other edges in P_{n_g}, the vertex degree difference is zero. Hence there exists 4 edges with degree difference one. Therefore, $Alb(P_{n_g}) = 4 = \sigma(P_{n_g})$.

Case 2: n is odd.

Let v_i be the central vertex in P_n and v_{i-1} and v_{i+1} be its neighbors. Then in P_{n_g}, $|deg(v_i) - deg(v_{i-1})| = 1$ and $|deg(v_i) - deg(v_{i+1})| = 1$. Also, similar to the previous case $|deg(v_2) - deg(v_1)| = 1$ and $|deg(v_n) - deg(v_{n-1})| = 1$. For all other edges in P_{n_g}, the vertex degree difference is zero. Hence $Alb(P_{n_g}) = 4 = \sigma(P_{n_g})$. □

Remark 0.3. *For a star graph S_n, $Alb(S_n) = (n-1)(n-2)$ and $\sigma(S_n) = (n-1)(n-2)^2$. But, its glue graph is K_{n-1}. Hence $Alb(S_{n_g}) = 0 = \sigma(S_{n_g})$*

Remark 0.4. *For a $r-$star graph, $S_{n_1,n_2,\ldots,n_r}$,*

$$Alb(S_{n_1,n_2,\ldots,n_r}) = \sum_{i=1}^{r-1} |n_i - n_{i+1}| + \sum_{i=1}^{r} n_i(n_i - 1)$$

$$\sigma(S_{n_1,n_2,\ldots,n_r}) = \sum_{i=1}^{r-1} (n_i - n_{i+1})^2 + \sum_{i=1}^{r} n_i(n_i - 1)^2.$$

Conclusion

In this work, we studied the Albertson index and Sigma index in some special graphs and derived graphs such as glue and shadow graphs. There are wide areas related to these indices that are yet to be explored like in derived graphs such as line graphs, edge-subdivision graphs and power of a graph, etc. Also, the study on inverse topological indices type problems has a wide scope for further research.

References

[1] M. O. Albertson, The irregularity of a graph, *In: Ars Combinatoria.* **46** (1997) 219–225.

[2] H. Aswathy and S. S. Sandhya, Even path decomposition of root square mean graphs, *AIP Conference Proceedings.* **2516** (01) (2022).

[3] R. Balakrishnan and K. Ranganathan, *A textbook of graph theory*, (Springer Science & Business Media, 2012).

[4] J. A. Gallian, A dynamics survey of graph labelling, *In Electronic Journal Of Combinatorics* 1. *Dynamic Surveys.* (2018), DS6.

[5] A. U Gunes et al., Inverse problem for Albertson irregularity index, *Turkish World Mathematical Society Journal of Applied and Engineering Mathematics.* **12** (02) (2022).

[6] I. Gutman and E. Ernesto, Topological indices based on the line graph of the molecular graph, *Journal of chemical information and computer sciences.* **36** (03) (1996) 541–543.

[7] I. Gutman et al, Inverse problem for sigma index, *MATCH Commun. Math. Comput. Chem..* **79** (02) (2018) 491–508.

[8] E. Murugan and G.R. Sivaprakash, On Domination number of a graph and its shadow graph, *Discrete Mathematics, Algorithms and Applications* **13** (06) (2021) p. 2150074.

[9] V. S. Shigehalli and G. V. Uppin, Some properties of glue graph, *Annals of Pure and Applied Mathematics* **6** (01) (2014) p. 98–103.

[10] D. B. West et al., *Introduction to graph theory*, (Prentice hall Upper Saddle River, 2001).